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Unsteady behaviour of a heterogeneous elastic beam floating on shallow water☆

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ABSTRACT

The unsteady behaviour of a thin elastic Euler beam with heterogeneous structural properties, floating freely on the surface of an ideal incompressible liquid is investigated using the linear theory. The unsteady behaviour of the beam is due to the incidence of a localized wave on its surface or initial deformation. Two methods of solving the problem are proposed in which the sagging of the beam is sought in the form of an expansion in eigenfunctions of the oscillations of a heterogeneous beam (the first method) or of a homogeneous beam (the second method) in the void. In both methods the problem is reduced to solving an infinite system of ordinary differential equations for the unknown amplitudes. The effect of different actions on a beam having a piecewise-constant distribution of the cylindrical stiffness and the specific mass is investigated. The eigenvalues of the systems of differential equations are determined.

Numerous results recently obtained on the hydroelastic behaviour of floating plates, which model, for example, ice sheets and floating platforms, mostly relate to the problem of the scattering of incident periodic surface waves (see, for example, the reviews Refs 1 and 2). To solve this problem, the flow of the liquid and the deformation of the elastic plate are assumed to be periodic functions of time. It is usually assumed that the plate is homogeneous in its structural properties, but in fact, both ice sheets and artificial structures are heterogeneous. In recent years there has been steady interest in investigating the effect of the heterogeneous properties of floating plates on their scattering properties (see, for example, Refs 3–5).

The solution of the unsteady problem of the behaviour of a floating elastic plate is considerably more complex. All the existing solutions of this problem were obtained for a homogeneous plate. The solution of the unsteady three-dimensional hydroelastic problem involves considerable computer costs even in the linear formulation. For example, the action of an unsteady load on a rectangular elastic plate floating on the surface of an infinitely deep liquid was considered in Ref. 6. Considerable simplifications can be made when investigating the behaviour of an elastic plate floating on shallow water. The unsteady behaviour of a circular plate was investigated in Ref. 7. A beam is often considered as an approximate model of an extended rectangular plate. The unsteady behaviour of the floating elastic beam was investigated for shallow water in Refs 8 and 9 and for an infinitely deep liquid in Ref. 10.

Below, we consider the following cases as examples of unsteady action: the incidence of a localized surface wave on a beam and initial deformation of the beam. These cases were investigated for a homogeneous beam floating on shallow water in Ref. 8, and in the case of the action of an arbitrary external load in Ref. 9.

1. Formulation of the problem

Suppose a continuous elastic beam of length L floats freely on the surface of a layer of an ideal incompressible liquid of depth H. The surface of the liquid not covered by the beam is free. The region S occupied by the liquid is split into three parts: $S_1(|x|<L)$, $S_2(x<-L)$, $S_3(x>L)$ where x is the horizontal coordinate. We will assume that the depth of the liquid is small compared with the wavelength of surface and flexural-gravitational waves, and we will use the shallow-water approximation. The velocity potentials, which describe the motion of the liquid in the regions S_j, are equal to $\varphi_j(x,t)$ $(j=1,2,3)$, where t is the time. The depth of the liquid under the beam is equal to $h=H-d$, where d is the sagging of the beam, which, for simplicity, will be assumed to be constant along the beam.

We will assume that a localized surface wave, the vertical displacement of the liquid in which is equal to $\eta_0(x,t)=f(x-\sqrt{gH}t)$ is incident on the beam from the left. The function $f(\xi)$ is only non-zero when $|\xi|<c$. Suppose that, at the instant of time $t=0$, the beam and

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the liquid are at rest in the regions S_1 and S_3, while a localized perturbation reaches the left edge of the beam in the region S_2. When $t > 0$, oscillations of the beam and the liquid begin in the region S_1, which give rise to wave perturbations diverging from the beam in regions S_2 and S_3.

The normal sagging of an inhomogeneous Euler beam $w(x,t)$ is described by the equation

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 w}{\partial x^2}\right) + m(x)\frac{\partial^2 w}{\partial t^2} + g\rho w + \rho\frac{\partial \varphi_1}{\partial t} = 0, \quad |x| \le L \tag{1.1}$$

where $D(x)$ is the cylindrical stiffness of the beam, $m(x)$ is its specific mass, ρ is the density of the liquid and g is the acceleration due to gravity. We will assume that the values of the functions $D(x)$ and $m(x)$ and their first derivative are piecewise-continuous and that these functions have integrable second derivatives. By Archimedes principle

$$d = \frac{1}{2\rho L}\int_{-L}^{L} m(x)dx \tag{1.2}$$

The following relation holds in linear shallow-water theory

$$\frac{\partial w}{\partial t} = -h\frac{\partial^2 \varphi_1}{\partial x^2}, \quad |x| \le L \tag{1.3}$$

In the regions outside the beam the velocity potentials satisfy the equations

$$\frac{\partial^2 \varphi_j}{\partial t^2} = gH\frac{\partial^2 \varphi_j}{\partial x^2}, \quad x \in S_j, \quad j = 2, 3 \tag{1.4}$$

The elevations of the free surface $\eta_2(x,t)$ and $\eta_3(x,t)$ in regions S_2 and S_3 respectively are found from the relations

$$\eta_j = -\frac{1}{g}\frac{\partial \varphi_j}{\partial t}, \quad x \in S_j, \quad j = 2, 3$$

At the edges of the beam we have the free-edge conditions, namely, the bending moment and the shearing force are equal to zero:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0, \quad |x| = L$$

In the liquid, when $|x| = L$ the conditions of continuity of the pressure and mass must also be satisfied:

$$\frac{\partial \varphi_1}{\partial t} = \frac{\partial \varphi_2}{\partial t}, \quad \frac{\partial \varphi_1}{\partial x} = \frac{H}{h}\frac{\partial \varphi_2}{\partial x}, \quad x = -L$$
$$\frac{\partial \varphi_1}{\partial t} = \frac{\partial \varphi_3}{\partial t}, \quad \frac{\partial \varphi_1}{\partial x} = \frac{H}{h}\frac{\partial \varphi_3}{\partial x}, \quad x = L \tag{1.5}$$

Far from the beam, there are no perturbations, and hence

$$\partial\varphi_2/\partial x \to 0, \quad x \to -\infty; \quad \partial\varphi_3/\partial x \to 0, \quad x \to \infty$$

The initial conditions have the form

$$w = \eta_3 = \varphi_1 = \varphi_3 = 0, \quad \eta_2 = \eta_0, \quad \varphi_2 = \varphi_0, \quad t = 0 \tag{1.6}$$

We will change to dimensionless variables, taking the length L as unity and the time $\sqrt{(L/g)}$ as unity. The following dimensionless functions are used

$$\delta(x) = \frac{D(x)}{\rho g L^4}, \quad \gamma(x) = \frac{m(x)}{\rho L}$$

2. Expansion in modes of the heterogeneous beam

We will seek the ragging of the beam in the form of an expansion in eigenfunctions of the oscillations of a heterogeneous beam with free ends in a void

$$w(x, t) = \sum_{n=0}^{\infty} a_n(t)\Psi_n(x) \tag{2.1}$$

The functions $a_n(t)$ are to be determined, while the functions $\Psi(x)$ are the solutions of the following eigenvalue problem in dimensionless variables

$$(\delta(x)\Psi_n'')'' = \mu_n^4\gamma(x)\Psi_n, \quad |x| \le 1; \quad \Psi_n'' = \Psi_n''' = 0, \quad |x| = 1 \tag{2.2}$$

The primes denote differentiation with respect to x. This problem is self-conjugate, as a consequence of which all the eigenvalues are non-negative: $\mu_n \ge 0$ ($n = 0, 1, 2, \ldots$). The eigenfunctions form a complete system that is orthogonal in a generalized sense, normalized as follows (everywhere henceforth, unless otherwise stated, the integration over x is carried out from -1 to 1):

$$\int \gamma(x)\Psi_k(x)\Psi_n(x)dx = \delta_{kn}$$

where δ_{kn} is the Kronecker delta.

Substituting expansions (2.1) into Eq. (1.1) and the initial conditions (1.6), multiplying the relations obtained by $\Psi_k(x)$ and integrating them over x from -1 to 1, we obtain the following system of ordinary differential equations

$$\ddot{a}_k + \sum_{n=0}^{\infty}(\mu_k^4\delta_{nk} + K_{nk})a_n + F_k(t) = 0, \quad a_n(0) = \dot{a}_n(0) = 0$$

where

$$K_{nk} = \int \Psi_n(x)\Psi_k(x)dx, \quad F_k(t) = \int \Psi_k \frac{\partial\varphi_1}{\partial t}dx$$

and the dot denotes differentiation with respect to time.

The solution for the potential $\varphi_1(x,t)$ will be sought in the form of the expansion

$$\varphi_1(x,t) = -\frac{1}{h}\left[V(t) + xU(t) + \sum_{k=1}^{\infty} Q_k(t)\sin\frac{k\pi}{2}(x+1)\right], \quad V(0) = U(0) = 0 \tag{2.3}$$

substituting which into Eq. (1.3), multiplying the relation obtained by $\sin[m\pi(x+1)/2]$ and integrating it over x from -1 to 1, we obtain

$$Q_m(t) = -\frac{4}{\pi^2 m^2}\sum_{n=0}^{\infty}\dot{a}_n(t)P_{nm}; \quad P_{nm} = \int \Psi_n(x)\sin\frac{m\pi}{2}(x+1)dx$$

The functions $V(t)$ and $U(t)$ are unknown and are found from the matching conditions (1.5).

We will now consider the behaviour on the solution in regions S_2 and S_3. We will seek the solution for the potential $\varphi_2(x,t)$ in the region S_2 in the form

$$\varphi_2(x,t) = \varphi_0(x,t) + \psi(x,t)$$

where $\varphi_0(x,t)$ is the potential of the incoming wave, which is found from the relation

$$\frac{\partial\varphi_0}{\partial x} = \frac{\eta_0}{\sqrt{H}}$$

The function $\psi(x,t)$ describes the velocity potential of the reflected wave. According to Eq. (1.4) the solution for the potential $\psi(x,t)$ can be sought in the form

$$\psi(x,t) = \begin{cases} A\left(\dfrac{x+1}{\sqrt{H}} + t\right), & -(1+\sqrt{H}t) < x < -1 \\ 0, & x < -(1+\sqrt{H}t) \end{cases} \tag{2.4}$$

We have a similar representation in the region S_3 for the function $\varphi_3(x,t)$, which describes the velocity potential of the transmitted wave

$$\varphi_3(x,t) = \begin{cases} B\left(t - \dfrac{x-1}{\sqrt{H}}\right), & 1 < x < 1+\sqrt{H}t \\ 0, & x > 1+\sqrt{H}t \end{cases} \tag{2.5}$$

Using matching conditions (1.5), we obtain the following differential equations for the functions $A(\xi)$ and $B(\xi)$

$$\dot{A} = \frac{1}{\sqrt{H}}\left(\frac{2}{\pi}\sum_{n=0}^{\infty}\dot{a}_n R_n - U\right) - \alpha(t), \quad \dot{B} = \frac{1}{\sqrt{H}}\left(U - \frac{2}{\pi}\sum_{n=0}^{\infty}\dot{a}_n Z_n\right) \tag{2.6}$$

with initial conditions

$$A(0) = B(0) = 0$$

where

$$\alpha(t) = \eta_0(-1,t), \quad R_n = \sum_{k=1}^{\infty}\frac{P_{nk}}{k}, \quad Z_n = \sum_{k=1}^{\infty}(-1)^k\frac{P_{nk}}{k}$$

Using the relations obtained, we have the following final system of ordinary differential equations for determining the oscillations of the beam in the form

$$\sum_{n=0}^{\infty}\left\{\left(\delta_{nm}+\frac{4}{\pi^2 h}T_{nm}\right)\ddot{a}_n - \frac{1}{\pi\sqrt{H}}[R_n(L_m+M_m)+Z_n(M_m-L_m)]\dot{a}_n + \right.$$

$$\left. + (\mu_m^4\delta_{nm}+K_{nm})a_n\right\} + \frac{M_m}{\sqrt{H}}U + \alpha(M_m-L_m) = 0$$

$$\dot{U} = -h\left\{\alpha + \frac{1}{\sqrt{H}}\left[U - \frac{1}{\pi}\sum_{n=0}^{\infty}(R_n+Z_n)\dot{a}_n\right]\right\} \tag{2.7}$$

where

$$T_{nm} = \sum_{k=1}^{\infty}\frac{P_{nk}P_{mk}}{k^2}, \quad L_m = \int\Psi_m(x)dx, \quad M_m = \int x\Psi_m(x)dx$$

After determining the functions $a_n(t)$ and $U(t)$ we can obtain all the characteristics of the motion of the liquid and the elastic beam. For example, for vertical elevations of the free surface in region S_2 we have

$$\eta_2(x,t) = \eta_0(x,t) + \zeta(x,t)$$

$$\zeta(x,t) = \begin{cases} -\dot{A}\left(\dfrac{x+1}{\sqrt{H}}+t\right), & -(1+\sqrt{H}t) < x < -1 \\ 0, & x < -(1+\sqrt{H}t) \end{cases}$$

while in region S_3

$$\eta_3(x,t) = \begin{cases} -\dot{B}\left(t-\dfrac{x-1}{\sqrt{H}}\right), & 1 < x < 1+\sqrt{H}t \\ 0, & x > 1+\sqrt{H}t \end{cases}$$

The functions $\dot{A}(\xi)$ and $\dot{B}(\xi)$ are found from Eqs. (2.6).

The solution of eigenvalue problem (2.2) can only be obtained fairly simply for piecewise-constant functions $\delta(x)$ and $\gamma(x)$. For more general cases it is necessary to use special numerical methods (see, for example, Ref. 11).

3. Expansion in modes of a homogeneous beam

The solution of the problem described in Section 1 can also be obtained by representing the sagging of the beam in the form of an expansion in eigenfunctions of the oscillations of a homogeneous beam with free ends in a void

$$w(x,t) = \sum_{n=0}^{\infty} b_n(t)W_n(x) \tag{3.1}$$

The functions $b_n(t)$ are to be determined, while the functions $W_n(x)$ are the solutions of the eigenvalue problem

$$W_n'''' = \lambda_n^4 W_n, \quad |x| \le 1; \quad W_n'' = W_n''' = 0, \quad |x| = 1$$

The solutions of this problem are well known and have the form

$$W_0 = 1/\sqrt{2}, \quad W_{2n} = D_{2n}[\cos(\lambda_{2n}x) + S_{2n}\operatorname{ch}(\lambda_{2n}x)]$$

$$W_1 = \sqrt{3/2}x, \quad W_{2n+1} = D_{2n+1}[\sin(\lambda_{2n+1}x) + S_{2n+1}\operatorname{sh}(\lambda_{2n+1}x)]$$

where

$$S_n = \cos\lambda_n/\operatorname{ch}\lambda_n, \quad D_n = 1/\sqrt{1+(-1)^n S_n^2}$$

The eigenvalues λ_n are found from the equation $\mathrm{tg}\lambda_n+(-1)^n\mathrm{th}\lambda_n=0$ for $n\geq 2$; $\lambda_0=\lambda_1=0$. The functions $W_n(x)$ form a complete orthogonal system, normalized as follows:

$$\int W_n(x)W_m(x)dx = \delta_{nm}$$

The solution for the potential $\varphi_1(x,t)$ is sought in the same form as when solving the unsteady problem for a homogeneous beam[9]

$$\varphi_1(x,t) = -\frac{1}{h}\left[\sum_{n=0}^{\infty}\Phi_n(x)\dot{b}_n + xu(t) + \upsilon(t)\right]$$

The functions $\Phi_n(x)$ satisfy the equation $\Phi_n''(x)=W_n(x)$. The functions $u(t)$ and $v(t)$ are found from matching conditions (1.5) and the representation of the solutions in the regions S_2 and S_3 by expressions (2.4) and (2.5) respectively. The differential equations for determining the functions $A(\xi)$ and $B(\xi)$ now have the form

$$\dot{A} = \frac{1}{\sqrt{H}}\left(\frac{1}{\sqrt{2}}\dot{b}_0 - \frac{\sqrt{3}}{2\sqrt{2}}\dot{b}_1 - u\right) - \alpha(t), \quad \dot{B} = \frac{1}{\sqrt{H}}\left(\frac{1}{\sqrt{2}}\dot{b}_0 + \frac{\sqrt{3}}{2\sqrt{2}}\dot{b}_1 + u\right)$$

The final system of ordinary differential equations takes the form

$$\sum_{n=0}^{\infty}\left[\left(X_{mn} - \frac{C_{mn}}{h}\right)\ddot{b}_n + (\delta_{mn} + Y_{mn})b_n\right] + \delta_{m0}\left(\frac{\ddot{b}_0}{2h} + \frac{\dot{b}_0}{\sqrt{H}}\right) +$$
$$+\delta_{m1}\left(\frac{\ddot{b}_1}{6h} + \frac{\dot{b}_1}{2\sqrt{H}} + \sqrt{\frac{2}{3H}}u\right) + \alpha\left(\sqrt{\frac{2}{3}}\delta_{m1} - \sqrt{2}\delta_{m0}\right) = 0$$
$$\dot{u} = -h\left(\alpha + \frac{\sqrt{3}}{2\sqrt{2H}}\dot{b}_1 + \frac{u}{\sqrt{H}}\right) - \frac{1}{2\sqrt{6}}\ddot{b}_1, \quad b_n(0) = \dot{b}_n(0) = u(0) = 0 \tag{3.2}$$

where

$$X_{mn} = \int\gamma(x)W_m(x)W_n(x)dx, \quad C_{mn} = \int W_m(x)\Phi_n(x)dx$$
$$Y_{mn} = \int\delta(x)W_m''(x)W_n''(x)dx$$

There are analytic expressions for C_{mn} in Ref. 9.

4. The energy relation

The total energy of the travelling surface wave before it encounters the floating beam is constant and equal to

$$E_0 = \int_{-(1+2c)}^{-1}\eta_0^2(x,0)dx$$

This energy is transmitted to the oscillations of the elastic beam and the scattered (transmitted and reflected) surface waves. When $t\to\infty$ the oscillations of the beam attenuate and it returns to its initial horizontal position. The energy of the reflected wave is

$$E_r(t) = \int_{-(1+\sqrt{H}t)}^{-1}\zeta^2(x,t)dx = \sqrt{H}\int_0^t\dot{A}^2(\xi)d\xi$$

while the energy of the transmitted wave

$$E_t(t) = \int_1^{1+\sqrt{H}t}\eta_3^2(x,t)dx = \sqrt{H}\int_0^t\dot{B}^2(\xi)d\xi$$

In this problem there is no energy dissipation and, consequently,

$$\lim_{t\to\infty}E(t) = E_0, \quad E(t) = E_r(t) + E_t(t)$$

The satisfaction of this equality can serve as a criterion of the accuracy of the method of solution employed.

5. Numerical results

To carry out numerical calculations we chose piecewise-constant values of the cylindrical stiffness and the specific mass in Eq. (1.1)

$$(\delta(x), \gamma(x)) = \begin{cases} (\delta_1, \gamma_1), & l < |x| < 1 \\ (\delta_2, \gamma_2), & |x| < l, \quad 0 < l < 1 \end{cases} \tag{5.1}$$

The sagging of such a beam, according to relation (1.2), in dimensionless variables is $d = \gamma_1(1-l) + \gamma_2 l$. The problem of the scattering of periodic surface waves for a specified heterogeneity of the beam was considered in Ref. 3.

To construct a solution of eigenvalue problem (2.2) it is useful to introduce the representation

$$\Psi_n(x) = \begin{cases} \Psi_n^+(x), & l < |x| < 1 \\ \Psi_n^-(x), & |x| < l \end{cases}$$

Problem (2.2) can then be written in the form of the equations

$$\Psi_n^{+''''} = \tau_n^4 \Psi_n^+, \quad l < |x| < 1; \quad \Psi_n^{-''''} = \sigma_n^4 \Psi_n^-, \quad |x| > 1$$
$$\tau_n = \chi_1 \mu_n, \quad \sigma_n = \chi_2 \mu_n, \quad \chi_j = (\gamma_j/\delta_j)^{1/4}, \quad j = 1, 2 \tag{5.2}$$

with the conditions of rigid adhesion at the points $|x| = l$

$$\Psi_n^+ = \Psi_n^-, \quad \Psi_n^{+'} = \Psi_n^{-'}, \quad \delta_1 \Psi_n^{+''} = \delta_2 \Psi_n^{-''}, \quad \delta_1 \Psi_n^{+'''} = \delta_2 \Psi_n^{-'''} \tag{5.3}$$

and the conditions of a free edge at the ends of the beam.

The solution of this problem can be split into even and odd components with respect to x.

For the even part of the solution of eigenvalue problem (5.2), (5.3) the eigenfunctions have the form ($k \geq 1$)

$$\Psi_0 = 1/\sqrt{2[\gamma_2 l + \gamma_1(1-l)]}, \quad \mu_0 = 0$$
$$\Psi_{2k}^- = \frac{B_{2k}}{2}(\Lambda_{2k} \cos \sigma_{2k} x + \Gamma_{2k} \operatorname{ch} \sigma_{2k} x)$$
$$\Psi_{2k}^+ = B_{2k}[\sin \theta_{2k} + \operatorname{sh} \theta_{2k} + D_{2k}(\cos \theta_{2k} + \operatorname{ch} \theta_{2k})], \quad \theta_n(x) = \tau_n(1-x)$$
$$\Lambda_{2k} = \frac{K_1^+ + D_{2k} K_2^+}{\cos z}, \quad \Gamma_{2k} = \frac{K_2^- - D_{2k} K_3^-}{\beta \operatorname{sh} z}, \quad D_{2k} = \frac{\beta K_1^- \operatorname{th} z + K_2^-}{K_3^- - \beta K_2^- \operatorname{th} z}$$
$$z = \sigma_{2k} l, \quad \beta = \frac{\chi_2}{\chi_1}$$
$$K_1^\pm = \varepsilon S^- \pm S^+, \quad K_2^\pm = \varepsilon C^- \pm C^+, \quad K_3^\pm = \varepsilon S^+ \pm S^-, \quad \varepsilon = \sqrt{\frac{\delta_1 \gamma_1}{\delta_2 \gamma_2}}$$
$$S^\pm = \sin y \pm \operatorname{sh} y, \quad C^\pm = \cos y \pm \operatorname{ch} y, \quad y = \theta_{2k}(l) \tag{5.4}$$

while the eigenvalues μ_{2k} are determined as the roots of the transcendental equation

$$(\beta K_1^- \operatorname{th} z + K_2^-)(K_3^+ \cos z + \beta K_2^+ \sin z) +$$
$$+ (K_2^+ \cos z - \beta K_1^+ \sin z)(\beta K_2^- \operatorname{th} z - K_3^-) = 0 \tag{5.5}$$

Table 1

	Versions			
n	I	II	III	IV
2	1.9887	1.7506	1.4066	1.0561
3	3.3019	2.8629	2.7062	1.9451
4	4.6231	4.1694	4.7054	3.1752
5	5.9440	5.1169	5.6181	4.2188
6	7.2648	6.2086	6.5915	4.8025
7	8.5857	7.6047	8.4265	5.4753
8	9.9066	8.6487	9.3648	6.6123
9	11.2275	9.6300	10.3300	7.7544

For the odd part of the solution the eigenfunctions have the form

$$\Psi_1 = x\sqrt{\frac{3}{2[\gamma_2 l^3 + \gamma_1(1-l^3)]}}, \quad \mu_1 = 0$$

$$\Psi_{2k+1}^{-} = \frac{B_{2k+1}}{2}(\Lambda_{2k+1}\sin\sigma_{2k+1}x + \Gamma_{2k+1}\operatorname{sh}\sigma_{2k+1}x), \quad k \ge 1$$

$$\Psi_{2k+1}^{+} = B_{2k+1}[\sin\theta_{2k+1} + \operatorname{sh}\theta_{2k+1} + D_{2k+1}(\cos\theta_{2k+1} + \operatorname{ch}\theta_{2k+1})]$$

$$\Lambda_{2k+1} = \frac{K_1^+ + D_{2k+1}K_2^+}{\sin z}, \quad \Gamma_{2k+1} = \frac{K_2^- - D_{2k+1}K_3^-}{\beta\operatorname{ch} z}, \quad D_{2k+1} = \frac{\beta K_1^- + K_2^-\operatorname{th} z}{K_3^-\operatorname{th} z - \beta K_2^-} \tag{5.6}$$

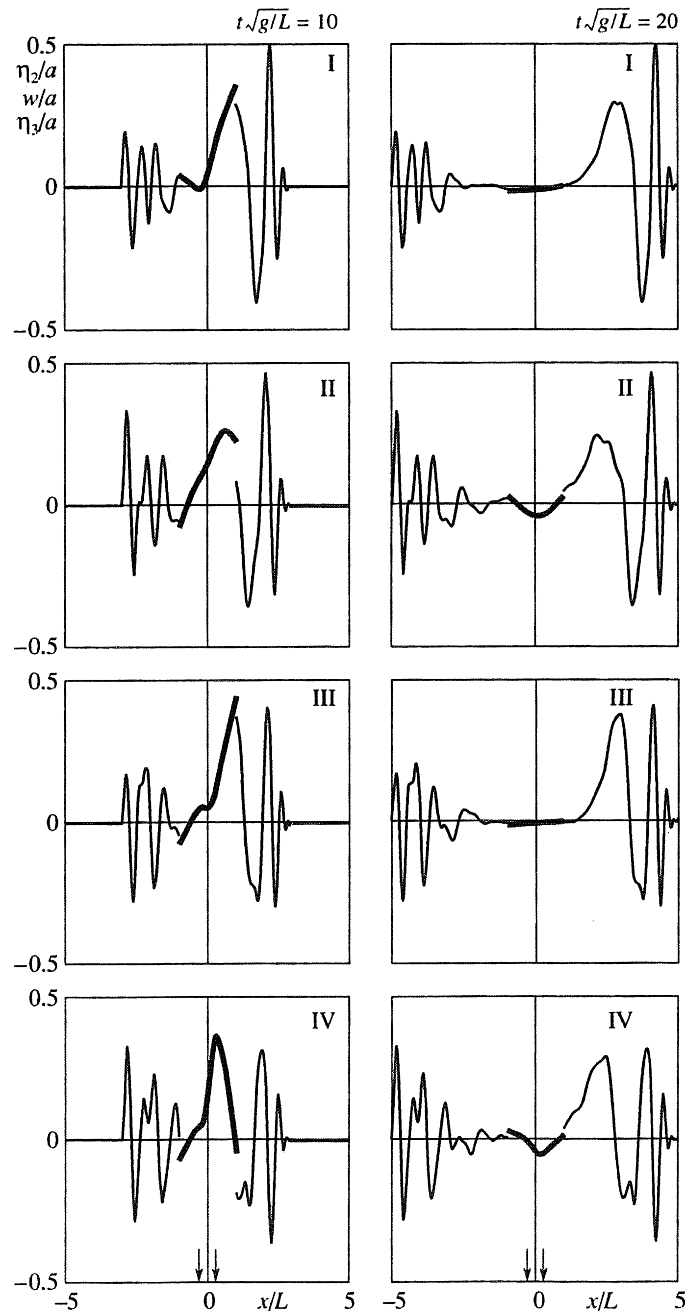

Fig. 1.

while the eigenvalues are found from the solution of the equation

$$(\beta K_1^- + K_2^- \mathrm{tg}z)(K_3^+ \sin z - \beta K_2^+ \cos z) + (\beta K_1^+ \cos z + K_2^+ \sin z)(\beta K_2^- - K_3^- \mathrm{th}z) = 0 \tag{5.7}$$

When calculating z and y for the odd modes the subscripts 2k must be replaced by 2k + 1.

The quantities B_{2k} and B_{2k+1} are found from the condition for normalizing the eigenfunctions (5.4) and (5.6), but they are not given here in view of their length. All the coefficients in Eqs. (2.7) and (3.2) are also calculated in analytic form.

In the calculations presented below, we used fixed parameters $\delta_1 = 5 \times 10^{-3}$ and $\gamma_1 = 10^{-2}$, while the values of δ_2, γ_2 and l were varied. Table 1 shows the eigenvalues μ_n (n = 2,. . ., 9) found from Eqs. (5.5) and (5.7) respectively for even and odd modes. A homogeneous beam was considered, for which $\delta_2 = \delta_1$ and $\gamma_2 = \gamma_1$ (Version I) and three versions of a heterogeneous beam with $l/L = 0.3$: $\delta_2 = \delta_1$ and $\gamma_2 = 5\gamma_1$ (Version II), $\delta_2 = \delta_1/10$ and $\gamma_2 = \gamma_1/5$ (Version III), and $\delta_2 = \delta_1/10$ and $\gamma_2 = 5\gamma_1$ (Version IV). It can be seen that the eigenvalues μ_n depend very much on the parameters of the heterogeneous beam. For a fixed value of δ_4 the eigenvalues decrease as the parameter γ_2 increases, but for a fixed value of γ_2 the eigenvalues increase as δ_2 increases.

The values of the initial dimensional parameters are $L = 500$ m, $H = 20$ m and $\rho = 10^3$ kg/m^3. The form of the localized surface wave, incident on the elastic beam, was chosen in the form

$$f(\xi) = \begin{cases} \dfrac{a}{2}\left(1 + \cos\dfrac{\pi\xi}{c}\right), & |\xi| < c \\ 0, \quad |\xi| > c \end{cases}; \quad \xi = x - t\sqrt{gH} - x_0$$

The total energy of this wave is constant with time and is equal to $E_0 = 3\rho g a^2 c/4$.

Using the reduction method we replace the infinite series in expansions (2.1), (2.3) and (3.1) by sums with a number of terms N_1, K and N_2 respectively. The systems of ordinary differential Eqs. (2.7) and (3.2) were solved numerically by a fourth-order Runge-Kutta method. In all the calculations $N_1 = 14$ and $K = 200$ for the first method and $N_2 = 40$ for the second method. Any further increase in these parameters has practically no effect on the results.

In Fig. 1 we show the behaviour of the free surface and the beam at instants of time $t\sqrt{g/L} = 10$ (the left column) and $t\sqrt{g/L} = 20$ (the right column) for $x_0/L = -1.25$ and $c/L = 0.25$ for Versions I, II, III and IV. For $x/L < -1$ we show the behaviour of η_2/a, for $|x|/L < 1$ we show the behaviour of w/a by the thick curves, and for $x/L > 1$ we show the behaviour of η_3/a. The difference between the solutions obtained by the first and second methods is very small and is only perceptible in the behaviour of the free surface for Version IV. The arrows indicate the position of the region $|x| \le l$ in which, for Versions II, III and IV, the parameters of the beam differ from the parameters of a homogeneous beam for Version I. When $t\sqrt{g/L} = 10$ the incident surface wave passes under the beam, which leads to deformations of the beam. Deflection of the beam, and also the behaviour of the free surface, depend considerably on the parameters of the structural heterogeneity.

When $t\sqrt{g/L} = 20$ a considerable part of the initial energy of the incident wave was transformed into the transmitted wave. However, the form of the transmitted wave differs considerably from the form of the initial wave and depends on the structural properties of the

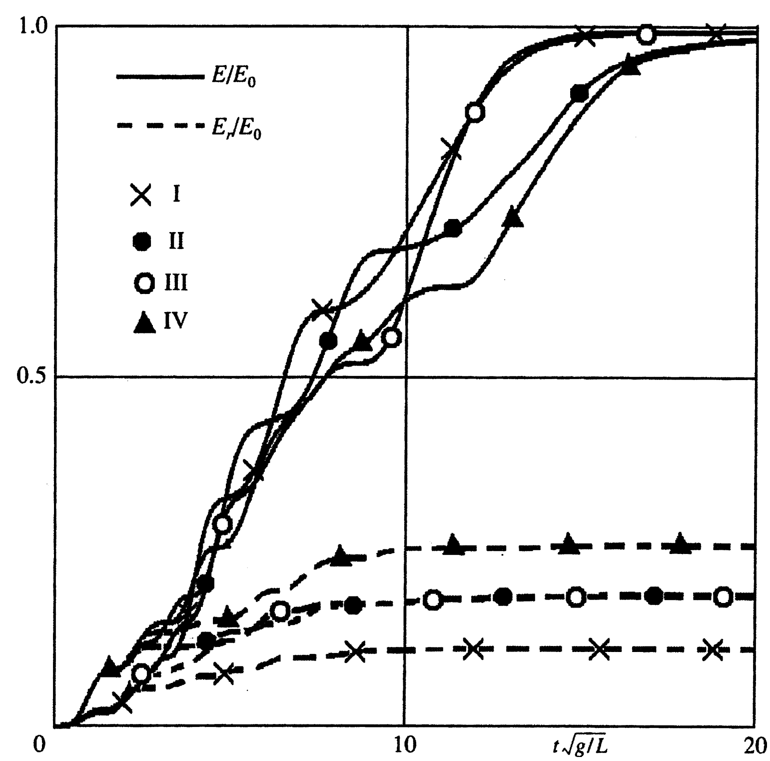

Fig. 2.

beam. For Versions I and III the oscillations of the beam practically ceased by this time, and for Versions II and IV they are still fairly considerable. These versions are characterized by a heavier middle part.

The change with time of the total energy of the transmitted and reflected waves $E(t)$, and also the total energy of the reflected wave $E_r(t)$, are represented in Fig. 2. The results of both methods are practically the same. The curves with the markers I – IV correspond to the number of the version. The continuous curves represent the results for the total energy $E(t)/E_0$, while the dashed curves are for the energy of the reflected wave $E_r(t)/E_0$. It can be seen that the limit value of the total energy is reached most rapidly for a homogeneous beam (Version I) and Version III, in which the middle part of the beam is less rigid and lighter. In this case the oscillations of the beam attenuate earlier than for Versions II and IV, in which the middle part is heavier (compare with Fig. 1). The most scattering into the reflected waves occurs for Version IV and amounts to about 25% of the initial energy of the incident wave, whereas for a homogeneous beam this value is approximately equal to 10%. An intermediate value (about 20%) for the reflected energy of the wave occurs in Versions II and III. It can be seen from Fig. 2 that the presence of structural heterogeneity has a considerable influence on the transformation of the incident wave.

We will also consider the unsteady behaviour of the beam when it has an initial deformation in a quiescent liquid. The initial conditions for this problem have the form

$$w(x,0) = w_0(x), \quad \eta_2(x,0) = \eta_3(x,0) = 0, \quad \varphi_j(x,0) = 0, \quad j = 1,2,3$$

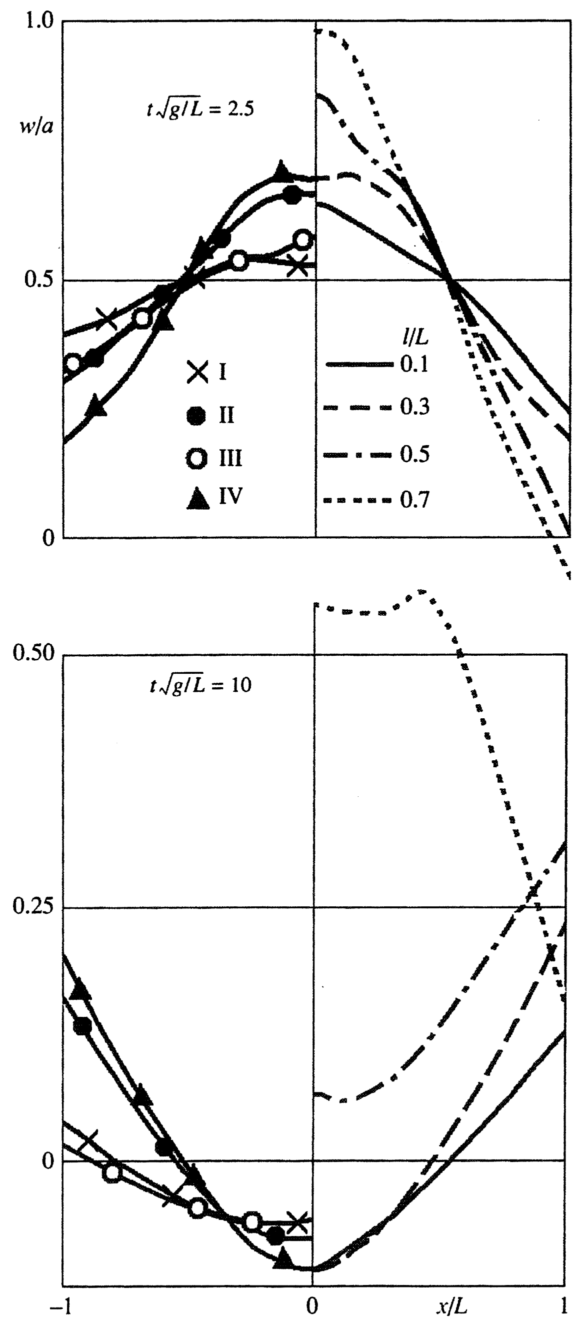

Fig. 3.

The function $w_0(x)$ is chosen in the form

$$w_0(x) = \frac{a}{2}\left(1 + \cos\frac{\pi x}{L}\right) \tag{5.8}$$

The systems of ordinary differential equations for determining the unsteady behaviour of the beam are identical with systems (2.7) and (3.2) respectively for the first and second methods, with the condition that now $\alpha(t) \equiv 0$. The initial conditions for $a_m(0)$ and $b_m(0)$ are given by the relations

$$a_m(0) = \int \gamma(x) w_0(x) \Psi_m(x) dx, \quad \dot{a}_m(0) = 0$$

$$b_m(0) = \int w_0(x) W_m(x) dx, \quad \dot{b}_m(0) = 0$$

The behaviour of the normal deflections of the beam is shown in Fig. 3 for $t\sqrt{g/L} = 2.5,\ 10$. For the chosen distributions of the initial deformation of the beam (5.8) and its structural heterogeneity (5.1), at all instants of time the normal deflections of the beam $w(x,t)$ are functions that are symmetrical about the origin of coordinates. In the left halves of Fig. 3 we show the results for Versions I–IV (see the description to Fig. 1), while in the right halves we show the results for $\delta_2 = \delta_1/10$ and $\gamma_2 = 5\gamma_1$ and different values of l. The dashed curves for $l/L = 0.3$ correspond to Version IV. It can be seen that the presence of structural heterogeneity of the beam has a considerable effect on its behaviour.

The oscillations of the beam attenuate with time, and when $t\sqrt{g/L} = 10$ they become fairly weak for Versions I and III, but for Versions II and IV, in which the middle part of the beam is heavier, they are still considerable. The effect of the stability of the oscillations when the middle part of the beam is heavier increases as the dimensions of this part increase for fixed values of δ_2 and γ_2.

In this problem it is also of interest to investigate the behaviour of the eigenvalues of systems of ordinary differential Eqs. (2.7) and (3.2). Thus, for example, after reduction we can write system (2.7) in the matrix form

$$\dot{\mathbf{Y}} = \mathbf{CY} + \mathbf{F}(t);\ \mathbf{Y} = \{a_0, a_1, \ldots, a_{N_1-1};\ \dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{N_1-1};\ U\}^T$$

where **C** is a square matrix of order $2N_1 + 1$ with constant elements, the vector $\mathbf{F}(t)$ is determined by the unsteady load, and the superscript T denotes transposition.

The eigenvalues and eigenvectors of the matrix **C** are often called "wet" modes, unlike the eigenvalues and eigenfunctions of problem (2.2), which are called "dry" modes. The properties of the "dry" modes are determined solely by the structural features of the beam, whereas the properties of the "wet" modes also depend on the properties of the liquid but do not depend on the type of unsteady load.

The eigenvalues of the matrix **C** were determined numerically. This matrix has one pure real eigenvalue ν_0 and $2N_1$ complex-conjugate eigenvalues $\nu_j (j = \pm 1, \pm 2, \ldots, \pm N_l)$, . The real parts of all the eigenvalues are negative. We will number the eigenvalues in the order in which their imaginary parts increase, i.e., $\mathrm{Im}\nu_j < \mathrm{Im}\nu_{j+1}$. The sign of the number j corresponds to the sign of the imaginary part of the eigenvalue.

The eigenvalues ν_j for $j = 0, 1, \ldots,$ J are shown in Fig. 4 for Versions I–IV. The value of J is found from the condition $Im(\nu_J\sqrt{L/g}) < 10$. The eigenvalues ν_j, determined for system (2.7), are shown by the dark circles and correspond to the solution of the problem by the first method, while the light circles are determined for system (3.2) and correspond to the second method. It can be seen that the disagreement between the eigenvalues obtained by the different methods are small and only become considerable for $j \geq 4$. The behaviour of the beam for large values of the time is mainly determined by the eigenvalues of the lower "wet" modes, since it is these that have the greatest real parts. It can be seen from Fig. 4 that, for the four versions considered, the eigenvalues for Versions II and IV have the greatest real parts, which also explains the occurrence of long-lived oscillations in these cases (compare Fig. 1 and Fig. 3).

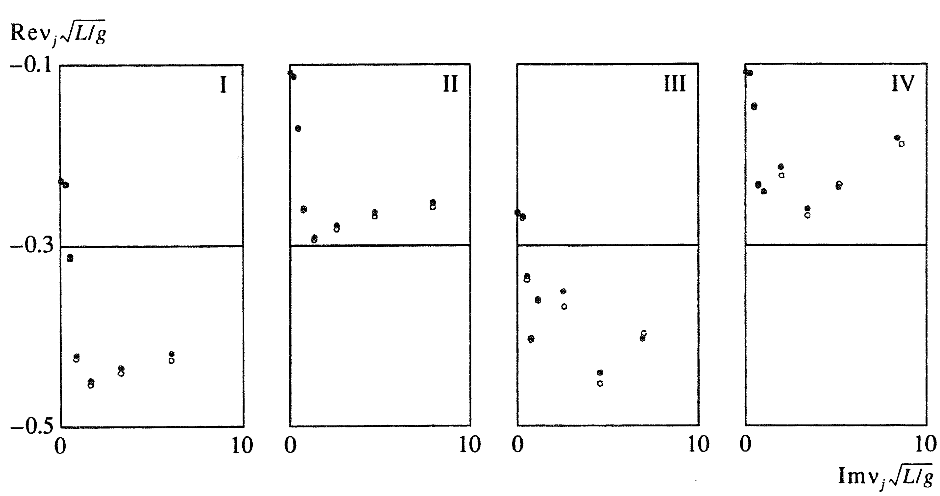

Fig. 4.

6. Conclusion

The results show that the structural heterogeneity of an elastic beam has a considerable effect on its unsteady behaviour. The presence of a heavier section in the middle part of the beam leads to the most prolonged oscillations, other conditions being equal. Using the results obtained earlier,[7,10] the methods proposed for solving the unsteady problem can be extended correspondingly to the case of heterogeneous circular plate, floating on shallow water, and a heterogeneous beam, floating on the surface of an infinitely deep liquid.

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